The Solar Swan Dive

Solutions for a Precalculus Physics Course

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The problem "if an object at the average distance of the Earth from the Sun could suddenly lose its (tangential) speed with respect to the Sun, how long would it take for it to crash into the Sun?" was originally presented to one of us (Dilsaver) by a colleague to whom a student had posed the question. This natural extension of the familiar uniform acceleration problems from high-school physics has come to be known as the "Solar Swan Dive" and leads to some interesting solutions.

Essentially all falling-object problems in high-school physics assume the distance fallen is insignificant compared with the distance to the center of attraction so that acceleration is constant. High-school students know that $g = 9.8\ \mathrm{m/s^2}$, even if they don't necessarily fully understand what that means. In the solar swan dive problem, however, acceleration is variable.

The first impulse for the teacher presented with such a question might be to tell the student the solution is beyond the scope of an introductory physics class and that he or she should ask again after a course or two in calculus. After all, the problem involves a second-order, nonlinear, differential equation! A second impulse might be for the teacher to think, "I can't work that problem and I'd better change the subject as quickly as possible!" But high-school physics is sufficient to solve the problem. In fact, there are a number of possible solutions that may be understood by students with no background in calculus. For completeness, a solution based on calculus is also included here. For simplicity, all solutions compute the time to reach the center of the Sun instead of the surface of the photosphere.

Method 1: Solution by Kepler's Third Law

Kepler's third law states that for a planet in an elliptical orbit, the square of the period, P, is directly proportional to the cube of the semimajor axis, a. If the semimajor axis is in astronomical units and the period is in Earth years, we have

$$P^2 = a^3 \tag{1}$$

With this relationship we can solve the solar swan dive problem as follows: Consider a family of elliptical orbits in which each ellipse has one focus at the Sun and the maximum Earth-to-Sun distance is fixed at one astronomical unit (A.U.). Ellipses in this family are distinguished by eccentricity.

The eccentricity, e, of an ellipse is the ratio c/a, where c is the distance from the center to either focus. With $c + a$ fixed at 1 A.U., we have

$$e = \frac{1-a}{a} \tag{2}$$

which rearranges to

$$a = \frac{1}{1+e} \tag{3}$$

Substituting Eq. (3) into Eq. (1) and solving for the period gives

$$P = \left(\frac{1}{1+e}\right)^{3/2} \tag{4}$$

As the eccentricity increases toward unity, the ellipse becomes almost a straight line between Earth and Sun. One-half the period of such an orbit becomes a close approximation to the time for a solar swan dive. Of course this is a limit process, but students without prior exposure to calculus seem to grasp the concept easily when presented with examples of the effect of varying eccentricity. Six confocal ellipses with eccentricities ranging from 0.0 to 0.999 are shown in Fig. 1. Table I gives a list of semiperiods (estimated "dive times") for these ellipses. (Figure 1 was drawn with a Strobe Model 100 digital plotter driven by an Apple II Plus microcomputer. Correspondence from readers with similar equipment is welcomed.)

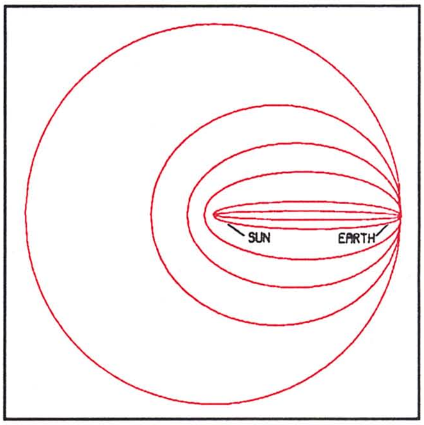

Fig. 1. Selected orbits approximating the solar swan dive.

If we consider the object falling into the Sun to be in a rectilinear "orbit" (i.e., a degenerate ellipse with unit eccentricity), which has semimajor axis equal to 0.5 A.U. and apply Eq. (4) we obtain

$$P = (0.5)^{3/2} \tag{5}$$

Thus $P/2$ gives a solar swan dive of $\frac{\sqrt{2}}{8}$ of an Earth year, or 64.6 days.

Method 2: Solution by Computer Approximation

Most high-school physics classes have access to some type of microcomputer. A program can easily be written to approximate the answer to the solar swan dive problem. For example, assume the acceleration due to the Sun's gravity is constant over a certain short

Table I. Half-periods of the orbits in Fig. 1.

Eccentricity of Ellipse	Half-Period (Days)
0.0	182.6
0.5	99.4
0.75	78.9
0.9	69.7
0.99	65.0
0.999	64.6

time, say one-tenth day. Calculate the distance fallen during that time increment, recompute the acceleration, and repeat the process until the Sun is reached.

This method, which is essentially a second-order Taylor series solution of an initial value problem,[1] will reinforce the variation in acceleration and allow students to see more clearly how this problem differs from their familiar falling body problems.

A computer solution in generic BASIC is given in Table II.

Method 3: Solution by Science Fiction

Many students (and teachers) enjoy reading science fiction. Occasionally a reference or an idea in a science-fiction story can generate some good discussion in a physics classroom. After solving the solar swan dive problem, we discovered that in Arthur C. Clarke's short story "Jupiter V" reference is made to "a well-known theorem stating that if a body falls from an orbit to the center of attraction, it will take point one seven seven of a period to make the drop."[2]

This theorem may be well known to someone well grounded in celestial mechanics, but perhaps is not to the average high-school physics teacher. Clarke's stories have a reputation for mathematical accuracy and the theorem mentioned gives the correct value of 64.566 days for the dive, since $\frac{\sqrt{2}}{8} = 0.17677....$

Method 4: Outline of a Solution by Calculus

Let x denote the distance of the falling object from the Sun at time t after the beginning of the fall. Then if we let M and G represent the mass of the Sun and the universal gravitation constant, respectively, we have the initial value problem

$$\frac{d^2x}{dt^2} = \frac{-GM}{x^2} \qquad (6)$$

with initial conditions

$$\frac{dx}{dt} = 0, x = a \text{ at } t = 0 \qquad (7)$$

If we multiply Eq. (6) by $\frac{dx}{dt}$ and integrate, we have

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{GM}{x} + C \qquad (8)$$

which on substitution of (7) and simplification becomes

$$\frac{dx}{dt} = -\sqrt{\frac{2MG}{a}}\sqrt{\frac{a-x}{x}} \qquad (9)$$

We note that the reciprocal of $\frac{dx}{dt}$ is $\frac{dt}{dx}$, so that

$$dt = -\sqrt{\frac{a}{2MG}}\frac{\sqrt{x}}{\sqrt{a-x}}dx \qquad (10)$$

Integrating by way of the trigonometric substitution $x = a\cos^2\theta$, we find that

$$t = \sqrt{\frac{a}{2MG}}\left(\sqrt{x(a-x)} + a \arccos\sqrt{\frac{x}{a}}\right) \qquad (11)$$

For the solar swan dive, $x = 0$, so

$$t = \frac{a\pi}{2}\sqrt{\frac{a}{2MG}} \qquad (12)$$

With $a = 1$ A.U., $t = 64.6$ days.

Conclusion

Many problems not generally considered in first-year physics classes are solvable by noncalculus physics. Showing students that they have the means to solve problems "not in the book" can increase their confidence and their willingness to attempt a variety of problem-solving methods. The authors believe that presenting a variety of methods of solution (or better, having students search for a variety of methods) may encourage creative thinking and better integration of the topics covered in an introductory physics course.

Table II. Sample computer solution of the solar swan dive.

```
10 REM ****************************************************************
20 REM **                 THE SOLAR SWAN DIVE                        **
30 REM **                                                            **
40 REM **               A TAYLOR METHOD SOLUTION                     **
50 REM ****************************************************************
60 REM
70 REM  M = MASS OF SUN, G = UNIVERSAL GRAVITATION CONSTANT
80 REM  X = DISTANCE OF DIVER FROM SUN, INITIALLY 1 A.U.
90 REM  V = VELOCITY, A = ACCELERATION, T = ELAPSED TIME
100 REM  DT = TIME INCREMENT OVER WHICH A IS ASSUMED CONSTANT
110 REM  ALL QUANTITIES IN MKS UNITS
120 REM
130 MG = 1.327E + 20
140 X = 1.495E + 11
150 V = 0 : T = 0
160 DT = 8640 : REM 0.1 DAY
170 T = T + DT
180 A = -MG/X/X
190 V = V + A * DT
200 X = X + V * DT + .5 * A * DT * DT
210 IF X > 0 THEN GOTO 170
220 LPRINT "IMPACT AT APPROXIMATELY " ; T/86400 ; " DAYS"
RUN
IMPACT AT APPROXIMATELY 64.4 DAYS
```


References

1. C.F. Gerald and P.O. Wheatley, *Applied Numerical Analysis*, 3rd ed. (Addison-Wesley, Reading, MA, 1984), pp. 301-303.
2. A.C. Clarke, "Jupiter V," *The Sentinel* (Berkley Books, New York, 1983), p. 120.