

A Journey into Triangular Number Land!

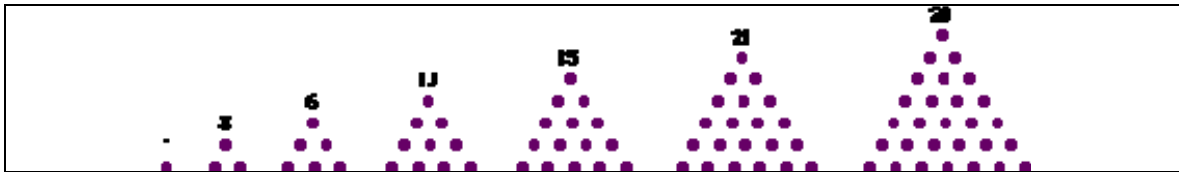
OHS Math Club!
*J Dilsaver**
**with some internet piracy!*

Triangular Numbers

Lots of boxes like this one have spaces for you to answer questions. Answer, and then check against Mr D's key.

A **triangular number** is an example of a figurate number: a figurate number is a number that can be represented by a regular geometric arrangement of points, such as a triangle, square, pentagon, etc. Triangular numbers also express the sums of the first n counting numbers. Note in the triangles below, each successive horizontal row of dots is one dot longer than the preceding row.

Triangular numbers can be thought of as the numbers of dots you need to make a triangle:



Triangular numbers appear in Pascal's triangle:

In the box at the right, write out the first several rows of Pascal's triangle (ask an older math club member for help if you're not sure how to do this.) Then, see if you can find the diagonals within Pascal's triangle which show the triangular numbers.

What interesting patterns are shown in the other diagonals of Pascal's triangle?

Some interesting properties of triangular numbers:

1. Let's denote the triangular numbers as follows:

$T_1, T_2, T_3, T_4, T_5, T_6, \dots$ and remember the triangular numbers are

1, 3, 6, 10, 15, 21, 28, \dots Can we find a way to calculate any particular triangular number without having to add the first n counting numbers? In other words, a

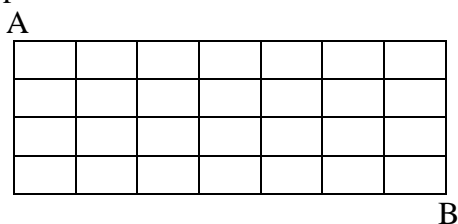
shortcut, or formula. For example, can we find T_{30} without having to add the first 30 counting numbers? Search for a pattern, and find how to do this.

Pascal's triangle

Search for a pattern:

Give a formula to find the n^{th} Δ number

2. **The Path Problem:** Consider the following problem. In the grid shown below find the number of distinct paths by which one can move from point A at the upper left corner of the grid, to point B at the lower right corner of the grid, provided one can only move down, or to the right. Try examining simpler cases to find a pattern. By this I mean, draw grids with fewer squares and simply count how many paths. A nice way to do this is to number the ways by which one can reach each intersection point.



Examine simpler cases of the path problem.

If we label the number of paths to each intersection point in the above grid from A to B, we get the following numbers:

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210

What relationship exists between the grid at the left and the Pascal's triangle you drew earlier?

This next question could be the nearest one on this whole handout:

List formulas, line by line, which generate the numbers in grid just above.

Check answers near the end of this paper.

3. **Sum of triangular numbers**, sum of sum of triangular numbers, sum of sum of sum of triangular numbers, and so on. Look carefully at the path problem for answers to these sums.
4. **The Handshake problem**. This is a classic problem, and goes like this. Some particular number of people are in a room or are invited to a party, and everyone shakes hands with everyone else. For any particular number of people, find out the total number of handshakes. Examine some simple cases in the space below, derive a formula, and find the connection (!) between the handshake problem and triangular numbers.

Examine simple cases.

Handshake formula:

5. **Factorial numbers**. The factorial operation is represented by the “!” sign. A few examples:

$$5! = 1 * 2 * 3 * 4 * 5 = 120$$

$$2! = 1 * 2 = 2$$

$$1000! = 1 * 2 * 3 * 4 \dots * 999 * 1000 = \text{a very large number.}$$

Factorial numbers appear many places in mathematics. For example, in the path problem in # 2 above, if we represent the number of horizontal moves as h , and the number of vertical moves as v then the answer to the number of distinct paths from A to B in the given rectangular grid is:

$$\frac{(h+v)!}{h!v!} \quad \text{This is a wonderful formula! Can you derive this?}$$

Notice the interesting analogy between factorial numbers and triangular numbers. Let T_n be the n^{th} triangular number

$$n! = 1 * 2 * 3 * 4 * 5 * \dots * (n-1) * n$$

$$T_n = 1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n$$

6. Pairwise sums of triangular numbers.

A neat pattern is discovered when we add triangular numbers in a pairwise manner. Consider the following sums,

$$1 + 3 = \underline{\hspace{2cm}}$$

$$3 + 6 = \underline{\hspace{2cm}}$$

$$6 + 10 = \underline{\hspace{2cm}}$$

$$10 + 15 = \underline{\hspace{2cm}}$$

$$15 + 21 = \underline{\hspace{2cm}}$$

$$21 + 28 = \underline{\hspace{2cm}}$$

$$28 + 36 = \underline{\hspace{2cm}}$$

(a) What conjecture can we make about the sum of two consecutive triangular numbers?

(b) Prove the above conjecture!

[Underclassmen should collaborate with older students on this part!]

6.5 Give a geometric proof, using dots, that the above conjecture in 6(a) is true.

7. An interesting property shows up when we take the differences between the squares of consecutive triangular numbers!

$$T_n = 1, 3, 6, 10, 15, 21, 28, 36, \underline{\quad}, \underline{\quad}, \underline{\quad}$$

$$(T_n)^2 = \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}$$

$$(T_n)^2 - (T_{n-1})^2 = \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}, \underline{\quad}$$

What is the interesting pattern???

Conjecture

Proof!

8. What cool thing happens when we multiply each triangular number by 8?
Fill in the following table of values: Note T_n represents the n^{th} triangular number.

n	1	2	3	4	5	6	7	8
T_n								
$8T_n$								
$8T_n + 1$								

Conjecture what the rest of this formula should be: $8T_n + 1 = \underline{\quad}$

Now, prove it!

9. Perfect numbers. Define perfect numbers. Note that all perfect numbers are triangular numbers.

Define perfect numbers and list the first few here:

How triangular numbers occasionally get used on GPML contest problems:

Triangular numbers get used occasionally in GPML problems, especially relay round problems. In the relay round 3 students sit in a row, and each in turn passes the answer to their problem back to their team mate, who will need that answer as part of their work. The cryptic letters “TNYWR” stand for the phrase “The Number You Will Receive”. Consider the following two relay round sample problems. Can you solve each?

GPML Relay Sample

1. Find the 6th triangular number. Pass this answer back.
2. Let $n = \text{TNYWR}$. Find the n th triangular number. Pass this answer back.
3. Let $n = \text{TNYWR}$. Find the n th triangular number.

Answers:

- 1.
- 2.
- 3.

GPML Relay Sample

1. Find the 20th triangular number. Pass this answer back.
2. Let $n = \text{TNYWR}$. Express n in base 4. Pass this number back.
3. Let $n = \text{TNYWR}$. Let n be a base 4 number. Convert this number to base 10. Let this base 10 number = B . B is the sum of first N counting numbers. Find N .

Answers:

- 1.
- 2.
- 3.

The answers to the relay problems appear later in this paper.

Answers.

Here is what the Pascal's triangle should look like: Note the diagonals which give the **triangular numbers**

1,3,6,10,15,21,28,

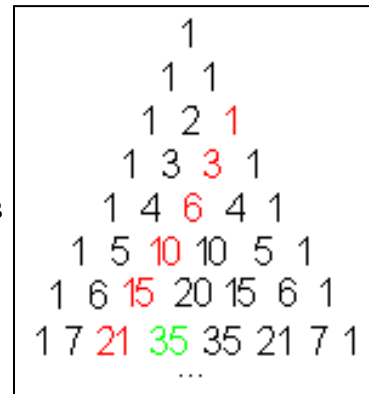
Can you find the fibonacci numbers hidden within Pascal's triangle?

What are Fibonacci numbers? _____

Answers to the relay problems:

First relay answers: 1. **21**, 2. **231**, 3. **26796**

Note that the first answer, 21, can be had by brute force by adding the first 21 natural numbers, but the second and third are just not obtainable in this manner in the 3 to 6 minute time limit of the GPML relay round



Second relay answers: 1. 210, 2. 3102₄, 3. 20

The Path Problem

<u>Numbers from Path Problem</u>	<u>Formula to generate the nth number</u>
1 1 1 1 1 1 1	$\frac{1}{1} = 1 = \frac{n^0}{0!}$
1 2 3 4 5 6 7	$\frac{n}{1} = \frac{n}{1!}$
1 3 6 10 15 21 28	$\frac{n(n+1)}{2} = \frac{n(n+1)}{2!}$
1 4 10 20 35 56 84	$\frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(n+2)}{3!}$
1 5 15 35 70 126 210	$\frac{n(n+1)(n+2)(n+3)}{24} = \frac{n(n+1)(n+2)(n+3)}{4!}$

There is a further shortcut way to give the formulas above, who can find it???

Factorial Powers:

$\overline{n^3} = n(n+1)(n+2)$ In the equation at the left, the exponent “3” is supposed to be overlined. This indicates rising factorial powers. Underling indicates falling factorial powers. This convenient, but not-often-seen-in-HS notation can be used to write the above patterns in a more succinct way.

$$\frac{\overline{n^0}}{0!} =$$

$$\frac{\overline{n^1}}{1!} =$$

$$\frac{\overline{n^2}}{2!} =$$

$$\frac{\overline{n^3}}{3!} =$$

$$\frac{\overline{n^4}}{4!} =$$

In general:

.

Extra Stuff

Ok, now let's consider another Δ of numbers analogous to pascal's Δ , but based on factorial powers.

First lets revisit Pascal's Δ , just for review:

$$\begin{aligned}(a+b)^0 &= 1 \\(a+b)^1 &= 1a + 1b \\(a+b)^2 &= 1a^2 + 2ab + 1b^2 \\(a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\(a+b)^4 &= \underline{\hspace{2cm}} \\(a+b)^5 &= \underline{\hspace{2cm}}\end{aligned}$$

List the coefficients in the form of Pascal's Δ

Note the recursion relation in Pascal's Δ , $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Now consider rising factorial powers:

$$\begin{aligned}x^{\bar{0}} &= x^0 = 1 \\x^{\bar{1}} &= x^1 = 1x \\x^{\bar{2}} &= x(x+1) = x^2+x \\x^{\bar{3}} &= x(x+1)(x+2) = x^3+3x^2+2x \\x^{\bar{4}} &= x(x+1)(x+2)(x+3) = 1x^4+6x^3+11x^2+6x\end{aligned}$$

						1
						1
					1	1
				1	3	2
			1	6	11	6
		1	10	35	50	24
	1	15	85	225	274	120

This triangle of numbers gives the coefficients in the expansion of $x^{\bar{m}}$, or x raised to any particular rising factorial power. Go back and look at the sums of the coefficients in any particular row of pascal's Δ , and then look at the sums of the coefficients of each particular row in this triangle. State these sums, row by row. By the way, these numbers in this triangle are known as Stirling numbers of the first kind. Can you find the recursion relationship to generate each new row of Stirling numbers?

Notation and use. The notation for stirling numbers of the first kind is not completely standardized, but often $\begin{bmatrix} n \\ k \end{bmatrix}$ is seen. Also, the triangle shown above is reversed left to right from the standard presentation. So $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ would be the row 6,11,6,1 in that order. Stirling numbers of the first kind count cycles. Consider 4 distinct objects, like these 4 letters: A,B,C,D. If we place these letters in a ring, we note that the orders, ABCD, BCDA, CDAB, and DABC are all the same (as we read from left to right, we encounter the same letters in the same order). But ACBD is different.

So, for A,BC,D and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ we have ABCD, ABDC, ACBD, ACDB, ADBC and ADCB. And thus $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 6$.

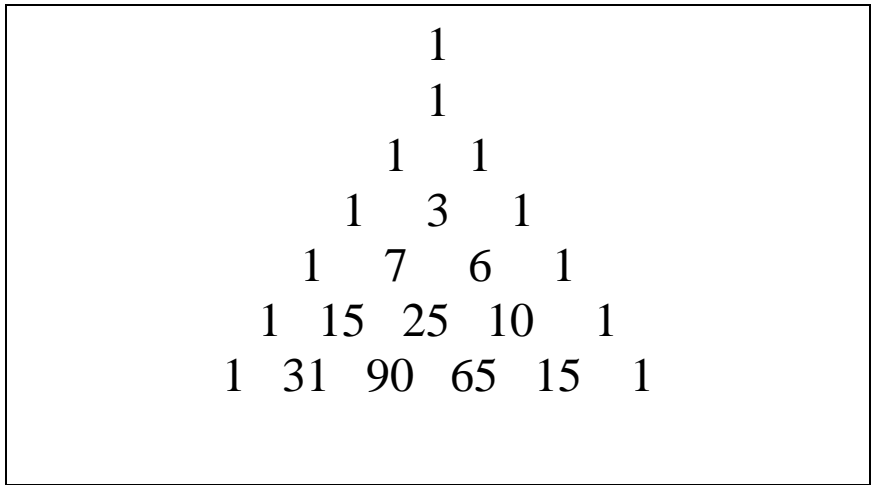
Now, lets consider $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

- ABC and D
- ACB and D
- ABD and C
- ADB and C
- ACD and B
- ADC and B
- BCD and A
- BDC and A
- AB and CD
- AC and BD
- AD and BC and thus $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$.

State the recursion relationship for stirling numbers of the first kind.

Are there Stirling numbers of the second kind (yes)

Consider this triangle:



These are the Stirling numbers of the second kind. Can you find a diagonal containing the triangular numbers in each of the two different Stirling number triangles? Can you find a recursion relation to generate successive rows of this triangle? (yes, there is such a relationship). Can you find any uses for these numbers (yes, there are uses)