The Method of Finite Differences.

1. Triangular pyramidal numbers can be formed by stacks of pennies, or some other object, where each layer forms a smaller triangle. The top layer contains 1 object, the next layer 3, and so on. These numbers form the sequence: 1, 4, 10, 20, 35, 56, 84, . . . Produce the formula $\dfrac{n(n+1)(n+2)}{6}$ using the method of finite differences.

Using the original sequence as line one, form rows of consecutive differences until reaching a constant difference, in this case that constant difference is 1.

1,		4,		10,		20,		35,		56,		84, . . .
	3		6		10		15		21		28	
		3		4		5		6		7		
			1		1		1		1			

For the next step we "back up" one diagonal further to the left, basing our reasoning on the fact that the third order differences are constant and in this particular case are equal to 1

0		1,		4,		10,		20,		35,		56,		84, . . .
	1		3		6		10		15		21		28	
		2		3		4		5		6		7		
			1		1		1		1		1			

Next, write the binomial combination, and simplify

$$t_n = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 1\binom{n}{3}$$

$$t_n = 0 + 1n + \frac{2n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}$$

$$t_n = \frac{0 + 6n + 6n(n-1) + n(n-1)(n-2)}{6}$$

$$t_n = \frac{6n + 6n^2 - 6n + n^3 - 3n^2 + 2n}{6}$$

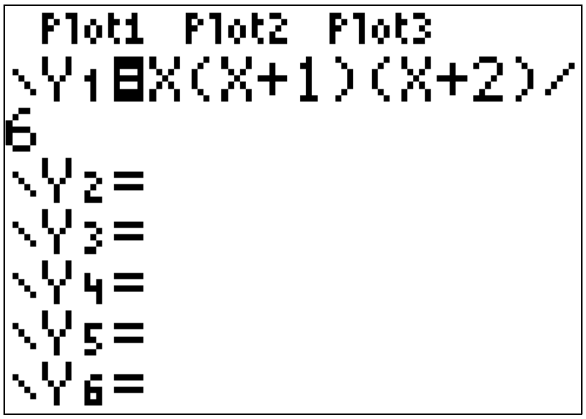
$$t_n = \frac{n^3 + 3n^2 + 2n}{6}$$

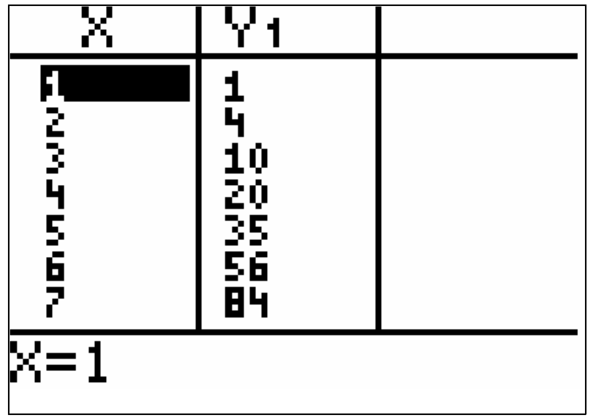
$$t_n = \frac{n(n+1)(n+2)}{6}$$

This is the required formula. It is easy (and fun) to quickly enter this formula into the TI-83/84 and using the table feature to see that it does in fact produce 1, 4, 10, 20, 35, 56, 84 for the integers 1,2,3, … 7

So, lets check a bit with the graphing calculator.

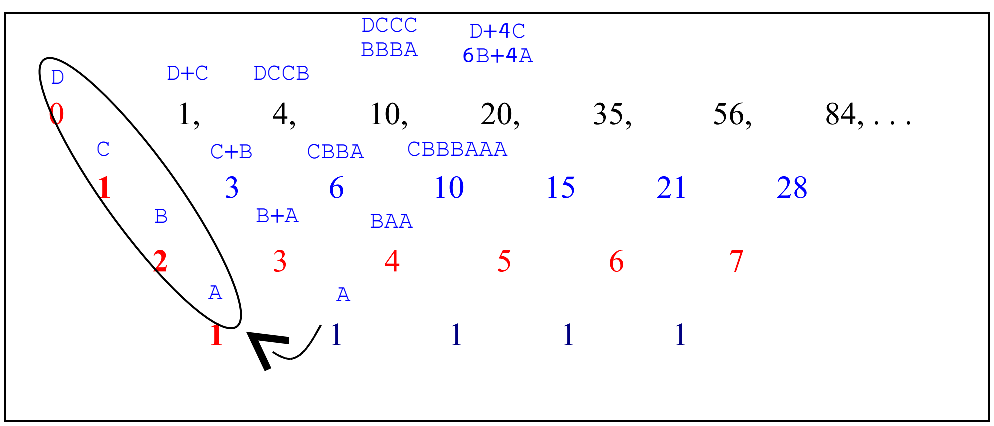
The screen captured at the left shows $t_n = \dfrac{n(n+1)(n+2)}{6}$ entered into the TI. Note that n is replaced with x, but otherwise the content is the same. The screen captured at the right shows the function values (Y values) for x inputs of the positive integers from 1 to 7. Note that the values given, namely 1, 4, 10, 20, 35, 56, 84 are exactly those we began with and we have indeed found an equation generating the values we desire.





So, why does this work? It's because of pascal's triangle and the binomial theorem. Lets use D, C, B, and A to represent the numbers in what I called the generating diagonal. This is the extra diagonal we inserted after forming the first order, second order, and third order differences. It's the diagonal in the oval in the diagram below. And then instead of actually performing the additions to generate new numbers in the table, just keep track of how many D's, C's, B's, and A's are used to form each entry.

Note that in the first row right above the 4, I have written DCCB, this is just a shortcut way to write D+C+C+B, and those letters come from the 1 and 3 to left of and below the 4. 1+3=4, and D+C + C+B are here represented by DCCB.



If you look carefully at the top row in this diagram perhaps you'll be reminded of pascal's triangle. Note that the 35 can be expressed as 0*1 + 1*5 + 2*10 + 1*10, and sure enough this is 35, and that could also be expressed as $0\binom{5}{0}+1\binom{5}{1}+2\binom{5}{2}+1\binom{5}{3}$, and those 5's in the combination notation are based on 35 being the 5th term. The general term is:

$$t_n = 0\binom{n}{0}+1\binom{n}{1}+2\binom{n}{2}+1\binom{n}{3}.$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

This method was originally developed by Newton, sometimes it is known as a Newton series. The general area of mathematics in which it resides is finite calculus, an area that was popular to study in universities about 1 century ago, but is out of favor today.

<u>Steps in the method of finite differences:</u>

1. The first line **<u>must</u>** be written as a **<u>sequence</u>** (this means commas, not plus or minus signs. Further, one can't just magically replace the plus signs by commas. Note that the problem of finding the sum of the squares of the consecutive integers could be written; $1+4+9+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$, but, if we find the first few values, say for n=1,2,3,…,7 we would write these as the sequence, 1,5,14,30,55,91,140, ... Even if the problem is stated as in the first equation, we must write the sequence shown before making a table of consecutive differences.
2. Compute the table of successive first order differences. Continue with second order differences, and so on. Stop when we produce a line where all the differences are constant. If a constant difference cannot be produced, double check for errors, but if indeed it cannot then the sequence is beyond the scope of this assignment. See Conway (p.) for more advanced methods. None of the problems on this paper require any more advanced methods.
3. Calculate the next previous diagonal to the left of the table which has been produced. I'll call this the generating diagonal.
4. Write the binomial combination and simplify.
5. This step is technically not required, but it is very quick to check your work using the y= screen and the TABLE function of the TI-83/84 calculator.

A few Problems:

1. Triangular pyramidal numbers can be formed by stacks of pennies, or some other object, where each layer forms a smaller triangle. The top layer contains 1 object, the next layer 3, and so on. These numbers form the sequence: 1, 4, 10, 20, 35, 56, 84, . . . Produce the formula $\frac{n(n+1)(n+2)}{6}$ using the method of finite differences.

2. Sums of consecutive integers, starting at 1 produce 1, 3, 6, 10, 15, 21, . . . which are known as the triangular numbers. These sums are given by the formula $S_n = \frac{n(t_1 + t_n)}{2}$ which simplifies to $S_n = \frac{n(n+1)}{2}$, start with 1, 3, 6, 10, 15, 21, . . . and produce this formula using the method of finite differences.

3. $\frac{n(n+1)(2n+1)}{6} = \sum_{i=1}^{n} i^2$

4. 1 + 3 + 5 + 7 + . . . +(2i-1) = i^2

5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 7 + ... + i(i+1) = \frac{n(n+1)(n = 2)}{3}$

6. (a) Find a polynomial which produces the values for the integers 1, 2, 3, …, 7

   6, 15, 22, 21, 6, -29, -90

   (b) What values does your polynomial give for 8, 9, and 10?

References:

Conway John H; Guy, Richard, *The Book of Numbers*, the chapter entitled “What Comes Next”

Gardner, Martin, The Colossal book of Mathematics.